Growth Optimal Portfolio Insurance and the Benefits of High Correlation

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Through a decomposition of the *growth rate* of the constant proportion portfolio insurance strategy (CPPI) this article unveils the rather surprising role that the correlation between the underlying assets plays on the performance of this type of investment strategy. The paper also introduces the growth optimal portfolio insurance strategy (GOPI), which combines the intuitively appealing objective of maximizing the value of the portfolio in the long run and the risk-management constraint of insuring a fixed proportion of the portfolio expressed in terms of the value of a given benchmark. The GOPI strategy tends to outperform the equivalent CPPI with the standard multiplier over long horizons though the level of the growth-optimal multiplier is often lower than the standard one. Hence the outperformance achieved by the GOPI does not necessarily come at the cost of a higher risk exposure.

The growth optimal portfolio (GOP) stands as an alternative to the utility maximization paradigm of portfolio choice since its discovery by Kelly (1956) and Latane (1959)[1]. The GOP has interesting theoretical properties such as outperforming any other portfolio in the long run in terms of wealth (Thorp (1971)) and minimizing the time to reach a given level of wealth (see Pestien and Sudderth (1985)). It also has the intuitive appealing of maximizing the expected geometric return mean and the median of wealth in the long run (see Ethier (2004)). These properties gained it the support from several authors, who believe that maximizing the growth rate is a reasonable investment objective for long horizon investors[2].

However, as Christensen et al. (2005) notes, "even if the GOP dominates another portfolio with a very high probability, the probability of the outcomes where the GOP performs poorly may still be unacceptable to an investor who is more risk averse than a log-utility investor". In other words, individual investors might have behavioral attitudes toward risk incompatible with the sole objective of maximizing the probability of attaining the highest possible level of wealth in the long run at all costs (the latter objective being represented by the GOP). Furthermore, in spite of their typically long horizons, institutional investors are often subject to short-term regulatory constraints such as limits to their underperformance of a given benchmark or on the ratio of their assets' value relative to the price of their liabilities (funding ratio constraints). A well-known strategy in investment management used to control such risk management constraints is known as Constant Proportion Portfolio Insurance (CPPI) (see Perold (1986), Black and Jones (1987), Perold and Sharpe (1988) and Black and Perold (1989)).

By maximizing the growth rate of the CPPI strategy this paper introduces the Growth Optimal Portfolio Insurance (GOPI). This asset allocation strategy combines the intuitively appealing objective of maximizing the growth rate of the portfolio and the need for a well-defined risk control relative to a benchmark. Our results suggest that the growth optimal strategy outper-

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forms the standard parametrization of the CPPI over long horizons.

Furthermore, our analysis of the growth rate of the CPPI strategy reveals that the "diversification benefits" that low correlations provide to fixed-mix long-only portfolios, are reversed for the portfolio insurance strategy: the higher the instantaneous correlation between the underlying assets, the higher the value of the portfolio insurance strategy, everything else being equal.

The positive effect of correlation in this type of investment strategy might be counter-intuitive for some readers due to a common confusion of this measure with the relative trend of assets' prices (see Lhabitant (2011) for a clear illustration of this misinterpretation). The intuition for this effect is that a higher level of correlation induces less relative return "reversals" between the two assets, hence diminishing what we call the "rebalancing drag" of the strategy. The importance and the role of correlation, which grows exponentially with the level of the multiplier of the strategy, is thoroughly illustrated with a graphical analysis.

Assets' Properties and Portfolio's Growth Rate

Former studies on the performance of rebalanced portfolios such as Booth and Fama (1992) and Fernholz (2002) focus on the impact that rebalancing has on the compounded return of unleveraged fixed-mix (also called constant-proportion) allocation strategies. In what follows we illustrate that the impact of volatilities, correlations and differences in expected returns among assets on the performance of portfolio insurance strategies is opposite with respect to their effect on unleveraged fixed-mix portfolios.

A first Intuition on the impact of Trends and Reversals on (Leveraged) Compounded Returns

One way to analyze the properties of the Constant Proportion Portfolio Insurance (CPPI) is to decompose its value into the Floor process and the Cushion process. Changes in the quotient of the values of the Performance-Seeking (PS) or risky asset, denoted S and the Reserve asset, denoted R, are naturally driven by the relative performance of the strategy's underlying assets. Black and Perold (1992) show that a return δ in the index ratio, i.e. $I(t) = S(t)/R(t)$, is equivalent to a proportional change in the Cushion magnified by m (the multiplier), $\Delta C/C = m\delta$. Hence, the dynamics of the Cushion are equivalent to the ones of a *leveraged* constant-proportion (CP) investment in the index ratio, with a leverage factor equal to m. Thus, the following example discusses some interesting properties of leveraged constant-proportion strategies.

Example:

In order to develop some intuition on the impact of trends and volatility in leveraged and buy-and-hold investments, consider the simplest case of a return series composed by two observations: $r = \{r_1, r_2\}$. Let, r_t take one of two values: u (up), for a positive return and $-d$ (down) for a negative one. An upward (downward) trend exists when $r_1 = r_2 = u$ ($r_1 = r_2 = -d$) and a "volatile" return reversal when $r_1 \neq r_2$. The compounded return of a constant-proportion investment with leverage factor "m", has the following properties (the same properties hold for a Buy-and-Hold investment with no leverage, i.e. $m = 1$):

- *Upward trend: The total return is greater than the sum*
  $$r^{1:2} = (1 + mu)^2 - 1 = 2um + (mu)^2$$
- *Downward trend: The total return is less negative than the sum*
  $$r^{1:2} = (1 - md)^2 - 1 = -2dm + (md)^2$$
- *"Volatile" returns: The total return is lower (or more negative) than the sum*
  $$r^{1:2} = (1+mu)(1-md)-1 = mu-md-udm^2$$

Hence, due to the compounding effect of returns, assets with stronger trends and/or lower "volatility", are better buy-and Hold investments, everything else equal. These effects are magnified with leverage.

In general, for a return series of T observations the compounded return of a buy-and-hold (BH) investment is equal to $r^{1:T} = \prod_{t=1}^{T}(1 +$

$r_t) - 1$, which is commonly expressed in terms of the geometric return average,

$$G = \left(\prod_{t=1}^{T} (1 + r_t) \right)^{\frac{1}{T}} - 1.$$

The following is a well known approximation for the geometric average that relates it to the arithmetic average and the variance of returns (see Booth and Fama (1992)):

$$G \approx A - \frac{1}{2} \frac{\sigma^2}{(1 + A)},$$

where A and σ^2 denote arithmetic average and variance of returns. Since A is usually small compared to 1, the approximation is sometimes done as (see Bernstein and Wilkinson (1997))

$$G \approx A - \frac{1}{2}\sigma^2. \tag{1}$$

Using the properties of the variance and the arithmetic average, it is straightforward to show that for leveraged investments the following similar relationship holds,

$$G^m \approx mA - \frac{1}{2} \frac{m^2\sigma^2}{(1 + mA)}. \tag{2}$$

Since the arithmetic average is a measure of "trend", equation (2) confirms the intuition shown in the two-observations example above for the case with multiple observations series: there is a volatility cost for holding an asset that is magnified by any present leverage. Conversely, positive "trends" have a positive impact on compounded returns and its effect is also magnified by leverage. Similarly, negative trends have less impact on the total return for a lower level of volatility, everything else equal. In other words, there is a tension between the "trend" and the "volatility" in the compounded return of an investment. This tension is affected by leverage, which gives proportionally more weight to the volatility cost than to trend gains in leveraged constant-proportion strategies.

For a buy-and-hold portfolio with two assets, the intuition follows through. A buy-and-hold strategy allocates proportionally more wealth with respect to the previous period to the asset presenting the highest return over the latest period. If in the next period occurs a relative return reversal, the strategy would have allocated more wealth to the relative loser asset. Conversely, if returns stay in the current trend (the latest winner asset is the winner asset the following period) the BH portfolio allocation to the winner asset would have been relatively higher, thus being a "winner strategy".

Another strategy of interest is the kind that maintains a fixed set of positive weights. A strategy aiming to keep a fixed proportion of the two assets needs to rebalance (trade) frequently as prices move. In order to keep a constant set of weights, the strategy needs to sell the latest relative winner asset and buy the relative loser one. For this reason, these types of strategies, namely unleveraged fixed-mix or constant-proportion portfolios, present a "buy-low and sell-high" behavior when return reversals occur (winner strategy) and a "buy-high and sell-low" one (loser strategy) in return trends. Thus, return trends and reversals have opposite effects on the compounded return of unleveraged fixed-mix with respect to buy-and-hold (and leveraged) strategies, due to their contrarian and trend-following allocation policies, respectively.

On Portfolio Rebalancing and its Growth Rate

This section discusses the concept of the *growth rate* of a portfolio and its one-to-one relationship with the portfolios' value over long horizons. We analyze the growth rate of buy-and-hold (BH) portfolios because their properties resemble those of portfolio insurance strategies, and the growth rate of fixed-mix (FM) portfolios because we use it later for deriving the growth rate of the portfolio insurance strategy.

Consider a portfolio defined by the vector of weights π, satisfying $\sum_i \pi_i = 1$, assigned to a constant set of n multiple assets. Throughout this section we assume continuous rebalancing and nil transaction costs.

In the simple Black-Scholes model for any risky asset A driven by a brownian motion W, its return is given by

$$dA(t)/A(t) = \mu dt + \sigma dW(t)$$

where μ is the asset's drift (also called trend or mean return) and σ is the volatility of the asset's returns. In this model, the explicit solution for the asset's price is

$$A(t) = A(0)e^{\gamma t + \sigma W(t)} \tag{3}$$

where $W(t) = \sqrt{t}z(t)$ for $z \sim \mathcal{N}(0,1)$. The term γ is called the "growth rate" of A because, for a long horizon, it is equal to the continuously compounded rate of return of the asset:

$$\gamma = \frac{1}{t}\ln\left(\frac{A(t)}{A(0)}\right) \text{ as } t \to \infty. \tag{4}$$

Hence, the growth rate is the continuously compounded rate of return but also the continuous time version of the geometric return average, because

$$\gamma = \frac{1}{t}E\left[\ln\left(\frac{A(t)}{A(0)}\right)\right]. \tag{5}$$

In fact, for assets following a geometric brownian motion, the growth rate, γ and the asset's drift μ are related as follows:

$$\gamma = \mu - \frac{1}{2}\sigma^2. \tag{6}$$

Interestingly, equation (6) is equivalent to the relationship between the geometric and arithmetic return averages of equation (1), derived in discrete time.

Fernholz and Shay (1982) show that the growth rate of a fixed-mix portfolio composed by n securities is given by

$$\gamma_\pi^{FM} = \sum_i^n \pi_i\gamma_i + \gamma_\pi^* \tag{7}$$

where,

$$\gamma_\pi^* = \frac{1}{2}\left(\sum_i^n \pi_i\sigma_i^2 - \sum_i^n\sum_j^n \pi_i\pi_j\rho_{ij}\sigma_i\sigma_j\right).$$

The first term on the right side of equation (7) is the weighted average of the growth rates of the component assets, and the second term γ_π^* is called the excess growth rate. Fernholz and Shay (1982) find that for **unleveraged** fixed-mix portfolios (i.e. $0 \le \pi \le 1$), the excess growth rate is always positive. This quantity is higher for higher standard deviations of the individual assets and for relatively lower or negative correlations. Thus this term is also called the "diversification bonus" (Bernstein and Wilkinson (1997), Booth and Fama (1992)). The intuition for the sign of the excess growth rate of unleveraged fixed-mix portfolios comes precisely from their "buy-low sell-high" behavior in the presence of return reversals discussed in the previous section[3].

On the other hand, the value process of a buy-and-hold portfolio is given by

$$V_\pi^{BH}(t) = \sum_i^n \pi_i(0)A_i(t)$$

where $\pi_i(0)$ is the proportion invested in asset i a time $t = 0$. Thus, the growth rate of the BH portfolio is:

$$\gamma_\pi^{BH} = \frac{1}{t}\ln\left(\sum_i^n \pi_i(0)\left(\frac{A_i(t)}{A_i(0)}\right)\right) \text{ as } t \to \infty.$$

From Jensen's inequality and the concavity of the log the following inequality holds,

$$\begin{aligned}\sum_i^n \pi_i\gamma_i &= \frac{1}{t}\sum_i^n \pi_i \ln\left(\frac{A_i(t)}{A_i(0)}\right) \\ &\le \frac{1}{t}\ln\sum_i^n \pi_i\left(\frac{A_i(t)}{A_i(0)}\right) \quad \forall \ 0 \le \pi \le 1.\end{aligned} \tag{8}$$

If the growth rates of the assets in the portfolio are equal then the inequality becomes an equality, making the unleveraged FM portfolio superior to the BH one. This is because there is no excess growth rate term on the BH's growth rate. Conversely, this implies that the BH portfolio presents an increase in return relative to the FM one for higher differences in the growth rates of the component assets (hence in stronger *trends* of the relative performance of the assets).

The geometric average and the value process are intrinsically related, since the former is a trivial transformation of the latter. For a set of n risky assets, A_i for $1 \leq i \leq n$, the value process of a fixed-mix portfolio can be expressed as: (see for instance Wise (1996), equation $A.3$)

$$V_\pi^{FM}(t) = V_0 e^{\gamma_\pi^* t} \prod_{i=1}^{n} \left(\frac{A_i(t)}{A_i(0)} \right)^{\pi_i}. \qquad (9)$$

Assuming a geometric brownian motion for each of the assets and replacing (3) in (9), the portfolio's value can be expressed in terms of the model parameters, i.e. expected returns, volatilities and correlations,

$$V_\pi^{FM}(t) = V_0 \exp\left(\gamma_\pi^* t + \sum_{i=1}^{n} \pi_i \gamma_i t + \pi_i \sigma_i W_i(t) \right) \qquad (10)$$

$$= V_0 \exp\left(\gamma_\pi^{FM} t + \sum_{i=1}^{n} \pi_i \sigma_i W_i(t) \right). \qquad (11)$$

Since the weighted average of brownian motions is a brownian motion, equation (11) matches the definition of the growth rate of equation (3) for the portfolio's value process.

Portfolio Insurance's Growth Rate and the Role of Correlation

Constant Proportion Portfolio Insurance (CPPI) asset allocation strategies split the portfolio between a reserve asset (R) and a performance-seeking asset (S), with a dynamic allocation to the risky asset defined by the product of the available risk budget or Cushion (C) at time t and a constant multiplier, m. The Cushion is the difference between the current value of the portfolio and the level of the Floor value to be insured, so the value of the portfolio is

$$V_m^{PI}(t) = F_t + C_t. \qquad (12)$$

The Floor guarantees a fixed proportion k of the value of the reserve asset at all times, as $F_t = kR_t$. Hence, as shown below, the Cushion can be interpreted as a leveraged fixed-mix portfolio that allocates $m\%$ to the performance-seeking asset and $(1-m)\%$ to the reserve asset. Consider the dynamics of the Cushion:

$$\begin{aligned} dC_t &= dV_t - dF_t \\ &= V_t \left(\frac{mC_t}{V_t} \frac{dS_t}{S_t} + \left(1 - \frac{mC_t}{V_t}\right) \frac{dR_t}{R_t} \right) \\ &\quad - F_t \frac{dR_t}{R_t} \\ &= mC_t \frac{dS_t}{S_t} + (1-m) C_t \frac{dR_t}{R_t} \\ &= C_t \left(m \frac{dS_t}{S_t} + (1-m) \frac{dR_t}{R_t} \right). \end{aligned}$$

Thus the return of the Cushion process is equal to the return of a fixed-mix portfolio with weights m and $(1-m)$.

Perold and Sharpe (1995) performed a graphical analysis of the payoff of standard CPPI portfolios as a function of the value of the risky asset. They studied a strategy allocating wealth between the riskless asset with constant interest rate and a risky asset. Using the fact that the Cushion process can be interpreted as a leveraged fixed-mix portfolio and equation (9), we can generalize their analysis for the case with a stochastically varying reserve asset. We focus on the more general case with a stochastic Reserve or "Core" asset because of its particular interest for pension funds, individual investors, and portfolio managers[4]. This introduces a new feature to the analysis: the role of the assets' correlation. For the two asset case the expression for the fixed mix portfolio value (9) simplifies to

$$V_\pi^{FM}(t) = V_0 \left(\frac{S(t)}{S(0)} \right)^{\pi} \left(\frac{R(t)}{R(0)} \right)^{1-\pi} e^{\gamma_\pi^* t}. \qquad (13)$$

Using the fact that the Cushion's dynamics are equivalent to a leveraged fixed-mix portfolio, we write the value of the portfolio insurance strategy, (12) as,

$$\begin{aligned} V_m^{PI}(t) &= kV_0 \left(\frac{R_t}{R_0} \right) \qquad (14) \\ &\quad + (1-k) V_0 \left(\frac{S_t}{S_0} \right)^m \left(\frac{R_t}{R_0} \right)^{1-m} e^{\gamma_m^* t}, \end{aligned}$$

where the Cushion's excess growth rate $\gamma_m^* = \frac{1}{2}m(1-m)(\sigma_S^2 + \sigma_R^2 - 2\rho\sigma_S\sigma_R)$ is in fact negative for $m > 1$, thus we call it a "rebalancing drag"[5]. Thus, contrary to the case of unleveraged fixed-mix strategies, volatility of component assets has a perversive effect on leveraged fixed-mix and portfolio insurance strategies, which confirms the intuition of the first section. More interestingly, the diversification benefits that unleveraged fixed-mix portfolios experience in the presence of low or negative correlation between assets is also reversed for portfolio insurance strategies (and for leveraged fixed-mix ones). A positive correlation decreases the rebalancing drag, therefore having a positive effect on compounded returns. The intuition for this is that a higher correlation induces fewer relative return reversals.

Using equations (13) and (14), we illustrate the important impact that correlation has in a portfolio insurance strategy's value. Figure 1 draws the portfolio's value corresponding to four correlation levels, i.e. $\rho = \{-0.5, 0, 0.5, 0.75\}$, and different combinations of values for $S = [50, 200]$ and $R = [80, 120]$ after 5 years (i.e. for a starting value of 100 dollars these values are equivalent to $[-13\%, 15\%]$ and $[-4\%, 4\%]$ return per annum respectively) with volatilities, $\sigma_S = 0.15$ and $\sigma_R = 0.05$. The black surface draws the end-of-period value of a CPPI strategy with $m = 4$ and $k = 0.9$ as given by equation (14). The red surface draws the end-of-period value of a Fixed-Mix Strategy as given by equation (13) with the same initial allocation to the risky asset: $m(1-k) = \pi = 0.4$. The upper panels of the figure correspond to negative and null correlation, i.e. $\rho = \{-0.5, 0\}$ and the lower panels to positive correlations, i.e. $\rho = \{0.5, 0.75\}$.

As observed in Figure 1, for a starting value of 100 dollars, the maximum possible value attained by the CPPI for $\rho = -0.5$ is of 189.9 dollars which is equivalent to 13.7% return per annum, while for $\rho = 0.75$, the CPPI reaches a value of 278.9, or 22.8% return per annum, everything else equal.

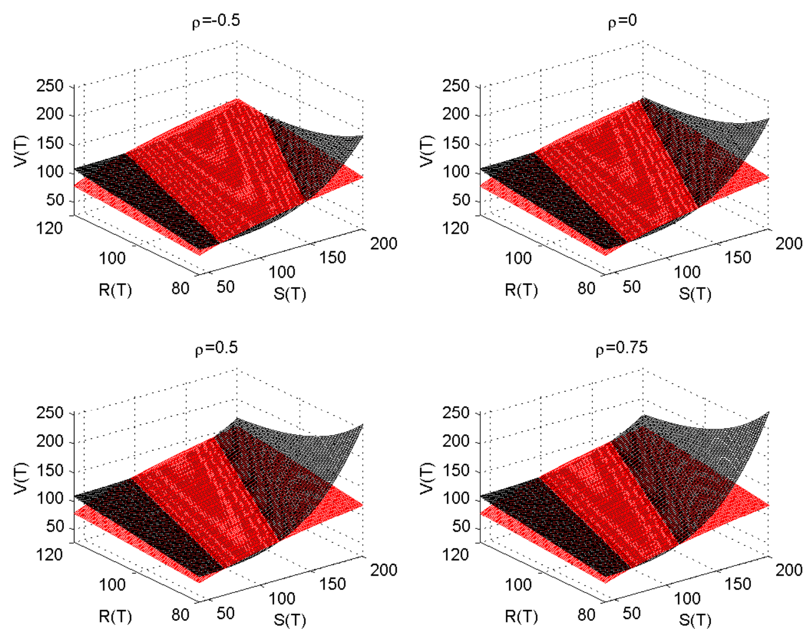

Figure 1: Impact of assets' correlation in Portfolio's value. The black surface draws the end-of-period value of a CPPI strategy as given by equation (14). The red surface draws the end-of-period value of a Fixed-Mix Strategy as given by equation (13).

On the other hand, the FM strategy presents a much more moderate change with correlation: For a starting value of 100 dollars, the maximum possible value attained by the FM strategy for $\rho = 0.75$ is 148.4 or 8.2% per annum, while for $\rho = -0.5$, the FM reaches a value of 150.1, or 8.5% per annum, everything else equal. Thus, the important gain obtained by the CPPI strategy with respect to the FM strategy, as observed in Figure 1 (area of black surface over red surface) as correlation increases, comes mostly from the increase in value of the CPPI strategy and not from the FM's value decrease.

In appendix A we show that the growth rate of the Portfolio Insurance strategy can be written as the weighted average of the growth rate of the Reserve asset and the growth rate of the Cushion plus a positive term,

$$\gamma_m^{PI} = k\gamma_R + (1-k)\gamma_m^{cushion} + \phi, \text{ with } \phi \geq 0. \tag{15}$$

The term ϕ tends to be small compared to $\gamma^{cushion}$ and γ_R. Furthermore, under mild assumptions, ϕ is an increasing function of $\gamma^{cushion}$ (see appendix A). Thus, for a given Reserve asset, the Cushion's growth rate should have a one-to-one relationship with the value of the portfolio in the long term.

Using expression (10), the portfolio insurance value can be written in terms of the assets' growth rates and covariances (equation (23) in appendix A), while the growth rate of the Cushion, $\gamma_m^{cushion}$ can also be written in terms of the assets' drifts and covariances (see appendix A for details):

$$\gamma_m^{cushion} = \gamma_m^* + m(\gamma_S - \gamma_R) + \gamma_R \tag{16}$$

$$\gamma_m^{cushion} = m(\mu_S - \mu_R) + \mu_R - \frac{1}{2}(m-1)^2\sigma_R^2 - \frac{1}{2}m^2\sigma_S^2 + m(m-1)\rho\sigma_S\sigma_R. \tag{17}$$

The term $m(\mu_S - \mu_R)$ in equation (17) illustrates the impact that the differences in assets' drifts (trends) have on the value of portfolio insurance strategies: the higher the expected outperformance of the performance-seeking asset (PS) with respect to the reserve asset (i.e. a stronger relative trend), the higher the expected return of the portfolio. Equation (17) also illustrates that assets with higher volatilities are less desirable for this type of strategy, everything else equal.

From equation (17) it is possible to infer the relative importance of the different assets characteristics on the Cushion's growth rate. In order of relative importance per unit of each term, for $m \geq 2$ we have[6]:

- Assets' covariance: $\rho\sigma_S\sigma_R$ (coeff: $+m(m-1)$)
- PS asset's variance: σ_S^2 (coeff: $-\frac{1}{2}m^2$)
- Expected outperformance: $\mu_S - \mu_R$ (coeff: $+m$)
- Reserve asset's variance: σ_R^2 (coeff: $-\frac{1}{2}(m-1)^2$)
- Reserve asset's drift (trend): μ_R (coeff: $+1$)

The order of importance presented above is indicative in particular of the Cushion growth rate's sensitivity to *changes* in the values of each of these assets' characteristics. However, the relative importance of these characteristics in the Cushion' growth rate not only depends on their coefficient but also on the actual values that the asset's parameters may take.

The theoretical decomposition of the CPPI's growth rate presented above uncovers the fact that the value of the portfolio insurance strategy increases with the correlation between its assets. We now turn to verify the relationship of the growth rate with the long term value of the portfolio insurance strategy with real data and leverage/short-selling constraints.

Empirical Test of Portfolio Insurance's Growth Rate

The theoretical characterization presented above implies a one-to-one relationship between CPPI's value and its Cushion's growth rate for a given Reserve asset over long horizons. However, the strategy's growth rate was derived under the assumptions of continuous rebalancing, unlimited leverage/short-selling and a simple geometric brownian motion model for the dynamics of the component assets. In this section we test to what extent this relationship holds in applications with real data and common leverage and short-selling constraints.

We use monthly returns of the following 13 EDHEC-Risk Alternative Investment indices[7] over the period January 1997 to March 2011:

- (ConvArb) Convertible Arbitrage
- (CTAs) CTA Global
- (Distress) Distressed Securities
- (EM) Emerging Markets
- (MNeutral) Market Neutral
- (EventD) Event Driven
- (FixArb) Fixed-Income Arbitrage
- (GMacro) Global Macro
- (LSequity) Long-Short Equity
- (MergArb) Merger Arbitrage
- (RelVal) Relative Value
- (ShortS) Short Selling
- (FoF) Fund of Funds

The summary statistics of this set of assets are presented in Table 1. These indices constitute a diverse set in which correlations across indices range from -76% to 93%, average returns between 2.4% and 10.4% and volatilities between 3% and 18.5%.

In order to verify if the theoretical relationship between growth rate and portfolio's value holds with real data, discrete rebalancing, and allocation constraints, we compare the ranking of the candidate performance-seeking assets according to the corresponding cushion's growth rate, the portfolio's value as given by equation (14), and the actual end-of-period value of the portfolio insurance strategy implemented with monthly data and allocation limits of $[0, 1]$ (i.e. short-selling and leverage are not allowed). The actual implementation of the CPPI strategy reallocates after a δ/m move in the exposure to the risky asset[8] with $\delta = 5\%$.

Notice that the theoretical value of the portfolio insurance strategy given by equation (14) is not just a transformation of the growth rate formula (15), as (14) uses the actual value of the underlying assets at the end of the period, instead of the value given by the brownian motion model. The estimation of the portfolio's value as given by equation (14) is denoted V_m while the actual implementation with exposure limits and discrete rebalancing of the Portfolio Insurance strategy is denoted PI_m. The difference in the ranking between V_m and PI_m can help us infer whether the differences observed in the ranking of the growth rate and the value of actual implementation PI_m come from the geometric brownian motion assumption or from the allocation constraints.

We also look at the ranking implied by the leveraged geometric return average G^m provided in equation (2) of the "index ratio" value (i.e. the ratio of the value of the performance-seeking asset and the reserve asset). This measure can be interpreted as a discrete time version of the CPPI Cushion's growth rate.

Although most former studies on the properties of the CPPI use a riskless Reserve asset, in general, the Reserve asset might vary in time stochastically, being for instance a portfolio hedging a stream of future consumption needs or replicating a given benchmark. Thus, we use the Fixed-Income Arbitrage index as the Reserve asset of the portfolio and rank the other 12 indices according to the Cushion's growth rate and to the end-of-period value of the portfolio as given by V_m and PI_m.

In order to have a reference to gauge the number of matches in the ranking among these criteria, we also look at the ranking of indices as implied by their simple geometric average return.

Table 1: Summary Statistics of Alternative Investment Indices. The upper panel presents the annualized arithmetic average, the standard deviation and the autocorrelation coefficient of order one of returns and the lower panel the cross-correlation matrix. The data are monthly returns over the period January 1997 to March 2011.

	ConvArb	CTAs	Distress	EM	MNeutral	EventD	FixArb	GMacro	LSequity	MergArb	RelVal	ShortS	FoF
μ	0.085	0.076	0.104	0.104	0.070	0.097	0.059	0.090	0.095	0.080	0.083	0.024	0.069
σ	0.067	0.086	0.063	0.129	0.030	0.062	0.048	0.057	0.076	0.037	0.045	0.186	0.061
AR(1)	0.586	0.023	0.509	0.323	0.273	0.388	0.508	0.052	0.265	0.311	0.454	0.141	0.332

	ConvArb	CTAs	Distress	EM	MNeutral	EventD	FixArb	GMacro	LSequity	MergArb	RelVal	ShortS	FoF
ConvArb	1.00	-0.04	0.72	0.57	0.49	0.71	0.78	0.39	0.57	0.55	0.85	-0.33	0.63
CTAs	-0.04	1.00	-0.04	0.02	0.18	-0.01	-0.00	0.51	0.09	-0.01	-0.02	0.11	0.15
Distress	0.72	-0.04	1.00	0.80	0.58	0.93	0.66	0.55	0.76	0.65	0.84	-0.60	0.82
EM	0.57	0.02	0.80	1.00	0.50	0.83	0.52	0.69	0.81	0.62	0.76	-0.69	0.86
MNeutral	0.49	0.18	0.58	0.50	1.00	0.63	0.41	0.56	0.64	0.55	0.62	-0.32	0.69
EventD	0.71	-0.01	0.93	0.83	0.63	1.00	0.59	0.62	0.88	0.84	0.90	-0.68	0.89
FixArb	0.78	-0.00	0.66	0.52	0.41	0.59	1.00	0.43	0.44	0.38	0.73	-0.17	0.54
GMacro	0.39	0.51	0.55	0.69	0.56	0.62	0.43	1.00	0.72	0.49	0.56	-0.42	0.79
LSequity	0.57	0.09	0.76	0.81	0.64	0.88	0.44	0.72	1.00	0.74	0.82	-0.76	0.92
MergArb	0.55	-0.01	0.65	0.62	0.55	0.84	0.38	0.49	0.74	1.00	0.75	-0.55	0.73
RelVal	0.85	-0.02	0.84	0.76	0.62	0.90	0.73	0.56	0.82	0.75	1.00	-0.55	0.81
ShortS	-0.33	0.11	-0.60	-0.69	-0.32	-0.68	-0.17	-0.42	-0.76	-0.55	-0.55	1.00	-0.66
FoF	0.63	0.15	0.82	0.86	0.69	0.89	0.54	0.79	0.92	0.73	0.81	-0.66	1.00

Table 2 presents the ranking of candidate assets according to the different criteria for three different multiplier values: $m = \{2, 3, 5\}$ and $k = 90\%$ (the proportion k does not affect the ranking among different methodologies).

The percentage of indices with exactly the same ranking according to the Cushion's growth rate and the end-of-period-value of the actual implementation of the strategy, is 100% (12 out of 12) for $m = 2$. For $m = 3$ the percentage is 83% (10 out of 12) and of 58% (7 out of 12) for $m = 5$. In this latter case, most differences in the ranking of the assets are not dramatic (for instance GMacro and LSequity simply swap their place).

Furthermore, the percentage of assets with the same ranking according to the growth rate $\gamma_m^{cushion}$, the value V_m and the leveraged geometric mean return of the index ratio G^m is 100% in all cases. This contrast with the percentage of rank matches using the simple geometric return average of each asset and the actual portfolio's value: 58%, 33% and 17% for $m = \{2, 3, 5\}$ respectively.

Table 2: Ranking of Performance-Seeking portfolios according to their geometric mean, the value of a PI portfolio rebalanced with exposure limits $\{0 \leq e \leq 1\}$, its theoretical value assuming continuous rebalancing and no short-settling limits (V_m) and the Cushion's growth rate for three different values of $m = \{2, 3, 5\}$. The Reserve asset is the Fixed-Income Arbitrage index.

Rank	Geometric Mean	$G^{m=2}$	$\gamma_{m=2}^{cushion}$	$V_{m=2}$	$DCS_{m=2}$
1	Distress	Distress	Distress	Distress	Distress
2	EM	EventD	EventD	EventD	EventD
3	EventD	LSequity	LSequity	LSequity	LSequity
4	LSequity	EM	EM	EM	EM
5	GMacro	GMacro	GMacro	GMacro	GMacro
6	RelVal	RelVal	RelVal	RelVal	RelVal
7	ConvArb	ConvArb	ConvArb	ConvArb	ConvArb
8	MergArb	MergArb	MergArb	MergArb	MergArb
9	CTAs	MNeutral	MNeutral	MNeutral	MNeutral
10	MNeutral	CTAs	CTAs	CTAs	CTAs
11	FoF	FoF	FoF	FoF	FoF
12	ShortS	ShortS	ShortS	ShortS	ShortS
Rank	Geometric Mean	$G^{m=3}$	$\gamma_{m=3}^{cushion}$	$V_{m=3}$	$DCS_{m=3}$
1	Distress	Distress	Distress	Distress	Distress
2	EM	EventD	EventD	EventD	EventD
3	EventD	LSequity	LSequity	LSequity	LSequity
4	LSequity	GMacro	GMacro	GMacro	GMacro
5	GMacro	EM	EM	EM	RelVal
6	RelVal	RelVal	RelVal	RelVal	EM
7	ConvArb	ConvArb	ConvArb	ConvArb	ConvArb
8	MergArb	MergArb	MergArb	MergArb	MergArb
9	CTAs	MNeutral	MNeutral	MNeutral	MNeutral
10	MNeutral	FoF	FoF	FoF	FoF
11	FoF	CTAs	CTAs	CTAs	CTAs
12	ShortS	ShortS	ShortS	ShortS	ShortS
Rank	Geometric Mean	$G^{m=5}$	$\gamma_{m=5}^{cushion}$	$V_{m=5}$	$DCS_{m=5}$
1	Distress	Distress	Distress	Distress	Distress
2	EM	EventD	EventD	EventD	EventD
3	EventD	GMacro	GMacro	GMacro	LSequity
4	LSequity	LSequity	LSequity	LSequity	GMacro
5	GMacro	RelVal	RelVal	RelVal	RelVal
6	RelVal	ConvArb	ConvArb	ConvArb	ConvArb
7	ConvArb	MergArb	MergArb	MergArb	MergArb
8	MergArb	EM	EM	EM	MNeutral
9	CTAs	MNeutral	MNeutral	MNeutral	FoF
10	MNeutral	FoF	FoF	FoF	EM
11	FoF	CTAs	CTAs	CTAs	CTAs
12	ShortS	ShortS	ShortS	ShortS	ShortS

These results imply that the differences with the actual implementation of the strategy mostly come from the portfolio' allocation constraints and not from the model assumptions used to derive the growth rate formula. The results also imply that the exposure limits are more likely to be reached with higher multipliers.

A similar ranking comparison exercise was performed using the 13 indices as candidate performance-seeking assets and the riskless asset with constant interest rate as the Reserve asset. The conclusions were very similar to the ones presented above. We do note report the results for space considerations (available upon request from the author).

This ranking exercise confirm that, in spite of the assumptions used to derive the theoretical value of the growth rate, its one-to-one relationship with the portfolio's value holds in most cases. Hence, if one would have good estimates of the expected volatilities, correlations and over-performance of a set of risky assets over the reserve asset, one could use the corresponding Cushion's growth rate estimate as a selection criterion of the performance-seeking asset. Our results also suggest that the Cushion's growth rate is a better criterion than the assets' return for selecting the CPPI's performance-seeking asset.

Introducing the Growth-Optimal Portfolio Insurance

A strand of the literature on portfolio selection argues in favor of "growth optimal portfolios" (GOP) (see for instance Markowitz (1976), Breiman (1961), Long et al. (1990)). GOP portfolios maximize expected log-utility of terminal wealth and the expected geometric return average and have the interesting theoretical property of outperforming all other strictly positive portfolios in long enough horizons (see Platen and Heath (2006) for a throughout analysis of the strategy).

Motivated by the absence of a risk dimension in the GOP, we propose instead to maximize the growth rate of the portfolio subject to the constraint of preventing its value to fall below a given fraction of the value of a benchmark (this fraction is chosen either by the investor or imposed by regulatory constraints). This is equivalent to maximize the growth rate of the classic CPPI strategy.

In former sections we document the close relationship between the long term value of portfolio insurance strategies and their growth rate. This result suggests that a reasonable objective for a long term investor with short-term constraints is to maximize the growth rate of the portfolio insurance strategy. Given a pair of assets with its corresponding parameter values and the fraction to be the guaranteed k, the multiplier should be chosen such that it maximizes the portfolio's growth rate.

Since the Floor process does not depend on the multiplier, maximizing the portfolios' growth rate with respect to the multiplier is equivalent to maximizing the growth rate of the Cushion (see appendix B for details). Taking the partial derivative of the Cushion's growth rate (in equation (16)) with respect to m equating to zero and solving for m yields the *growth optimal* multiplier:

$$m^* = \frac{\gamma_S - \gamma_R + \gamma^*}{2\gamma^*} \tag{18}$$

where $\gamma^* = \frac{1}{2}(\sigma_S^2 + \sigma_R^2 - 2\rho\sigma_S\sigma_R)$.

In order to illustrate the importance of the choice of the multiplier value, Figure 2 plots the Cushion's growth rate as given by equation (16) for the following parameter values: $\mu_S = 0.08$, $\sigma_S = 0.15$, $\mu_R = 0.03$, $\sigma_R = 0.05$, all possible values for the correlation coefficient, i.e. $\rho = [-1, 1]$, and different multiplier values, i.e. $m = [1, 10]$.

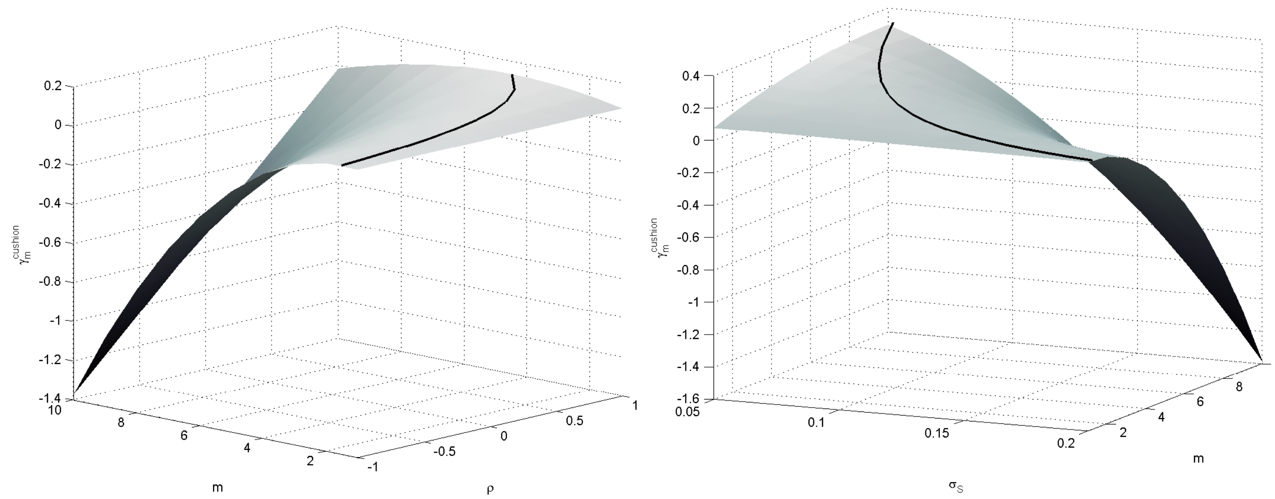

Figure 2: On the left panel, The surface represents the Cushion's growth rate for different multiplier values and correlations. On the right panel, the surface represents the Cushion's growth rate for different multiplier values and volatilities. The black line represents the value of the Cushion's growth rate for the optimal multiplier values corresponding to the different levels of the correlation coefficient ρ (left panel) or the Performance-Seeking asset's volatility σ_S (right panel).

The left panel of Figure 2 illustrates that m^* (dark line) maximizes the Cushion's growth rate of the strategy for every level of correlation. This figure also illustrates two interesting interactions between the correlation and the multiplier: *i*) the shape of the gray surface indicates that for uncorrelated or negatively correlated assets, the choice of the multiplier becomes critical in determining the Cushion's growth rate, ceteris paribus and *ii*) the optimal multiplier (dark line) increases with the correlation between the two assets, everything else being equal. Similarly, the right panel of Figure 2 illustrates that for highly volatile Performance-Seeking assets, the choice of the multiplier becomes critical (gray surface) and the growth-optimal multiplier (dark line) decreases fast with the PS asset's volatility, everything else being equal.

It is well-documented that expected returns, volatilities and correlations for most assets present significant variations in time. If any of these values varies in time, the multiplier should be adjusted accordingly in order to maintain the optimality condition:

$$m_t^* = \frac{\gamma_S(t) - \gamma_R(t) + \gamma^*(t)}{2\gamma^*(t)}. \qquad (19)$$

In the particular case with a riskless asset as the reserve asset with instantaneous risk-free rate $\mu_R = r$, with $\sigma_R = 0$, equation (19) yields

$$m_t^* = \frac{\mu_S(t) - r}{\sigma_S^2(t)}. \qquad (20)$$

Using the standard utility maximization setting, Merton (1971), Grossman and Vila (1992) and Basak (2002) find a remarkably similar solution for an optimal multiplier in the case of a portfolio insurer investor with CRRA preferences[9]. Grossman and Vila (1992)'s optimal multiplier is equal to (20) times the inverse of the investor's risk aversion parameter. From a practical standpoint, the problem with this approach is precisely to select the risk aversion parameter value, for which there is no general consensus. On the other hand, the growth-optimal multiplier does not need a risk aversion parameter. Hence there is no ambiguity about its optimal (unique) value[10].

Maximum Multiplier

The multiplier, m is a key parameter that determines the behavior of the portfolio insurance strategy. Perhaps the most common way to determine the multiplier of the CPPI in practice is to use the maximum value that would allow the Cushion to remain positive even in the "worst case scenario". In discrete time, in order to insure that the Cushion remains positive, the multiplier has to satisfy the following condition:

$$\frac{C_{t+1}}{C_t} = m\frac{S_{t+1}}{S_t} + (1-m)\frac{R_{t+1}}{R_t} > 0$$
$$\Leftrightarrow \frac{S_{t+1}}{S_t} > \frac{(m-1)}{m}\frac{R_{t+1}}{R_t}$$

or equivalently

$$\begin{aligned} mr_S(t,t+1) + (1-m)r_R(t,t+1) \geq -1 \\ m(r_S(t,t+1) - r_R(t,t+1)) \\ \geq -(1 + r_R(t,t+1)). \end{aligned} \tag{21}$$

For $(r_S(t,t+1) - r_R(t,t+1)) < 0$, the inequality (21) gets inverted,

$$m \leq \frac{-(1 + r_R(t,t+1))}{r_S(t,t+1) - r_R(t,t+1)},$$

where t and $t+1$ are any two portfolio rebalancing dates. Hence, the maximum value for the multiplier that could guarantee in general the Cushion's positivity condition (21) is:

$$m = \frac{-(1 + min(r_R))}{min(r_S - r_R)}. \tag{22}$$

In the particular case where the reserve asset is the riskless one, r_R has a minimum value equal to 0 (assuming positive interest rates) and its values are usually small compared to the magnitude of the returns on stocks. For this reason, condition (22) is very commonly approximated by:

$$m = \frac{-1}{min(r_S)}.$$

Although this value of the multiplier would respect the insurance promise of the portfolio strategy, complying with a feasibility condition is not necessarily the optimal choice from an investor's perspective. As we saw in previous sections, the rebalancing drag increases with the multiplier at a rate of $m(m-1)$. In the next section we compare the growth optimal portfolio insurance we introduced, with the equivalent portfolio insurance strategy defined by the aforementioned methodology to determine the multiplier.

Empirical test of the Growth-Optimal Portfolio Insurance strategy

In order to empirically evaluate the performance of the GOPI strategy we use the standard set of assets of the CPPI. Using monthly data of the T-bill rates as the reserve asset and the value-weighted CRSP broad market stock index[11] as the risky asset from January 1926 until December 2010, we compare the performance of the GOPI and the CPPI strategies (the latter being the standard portfolio insurance strategy using the multiplier as given by equation (22)). Both portfolio-insurance strategies have the constraint of preserving 90% of the value of the reserve asset.

First, we perform an in-sample backtest in which we use the entire sample to estimate the optimal and the maximal multipliers (for the GOPI and the standard CPPI). The second test is an out-of-sample exercise in which we use the first half of the period as a calibration sample. This backtest is performed over the second half of the sample using only the data available at every point in time, as if an investor would have implemented the strategy in real time. For the GOPI strategy, we implement two versions of the optimal multiplier: a static version using sample estimates of expected returns, volatilities, and correlation from returns in the calibration period, and a "dynamic" version of the optimal multiplier. In order to estimate the dynamic optimal multiplier series, we fit[12] a Dynamic Conditional Correlation model (Engle and Sheppard (2001)) and use the estimated parameters available at each point in time to infer the variance and covariance over the following month. We use a 10 year moving average to estimate the expected return of the stock index, and use the latest available interest rate as its next period forecast[13].

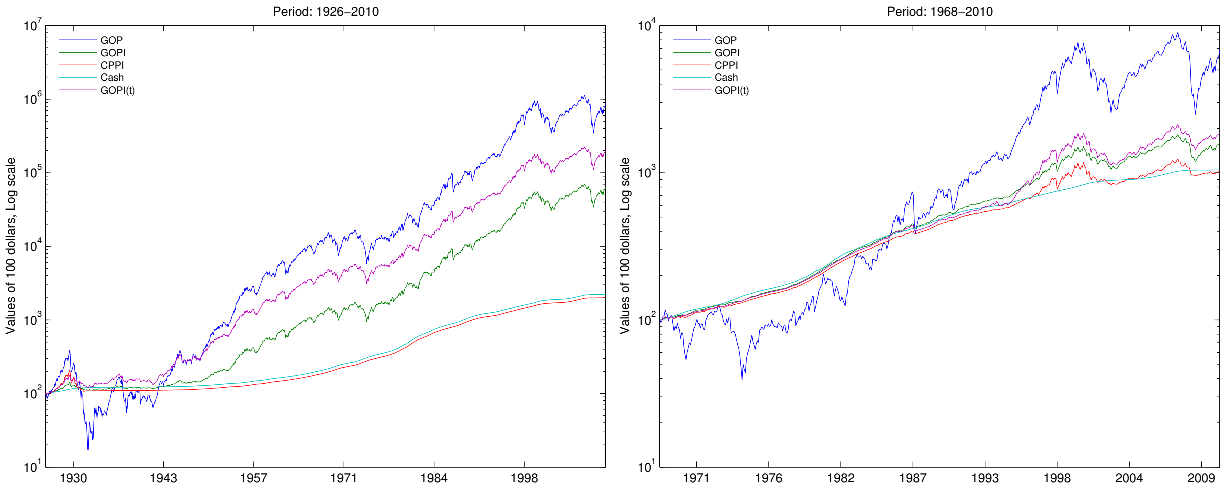

Figure 3: The left panel presents the in-sample Backtest in which all parameters were estimated using the entire sample period (1926:01-2010:12). The right panel presents the out-of-sample test in which all parameters are estimated using the first half of the sample (1926:01-1968:05) and data available at each point in time. The test is performed over the second half of the sample (1968:06-2010:12).

We also provide comparison with the GOP or Kelly criterion[14]. The allocation of the GOP to the risky asset in the case with two assets is equal to the multiplier of the GOPI. In the first Backtest we use the entire sample to estimate all the parameters that determine the optimal and the standard multipliers and the allocation of the GOP. The left panel of Figure 3 presents the cumulative performance of 100 dollars invested at the beginning of the period in each of the four strategies (GOP, GOPI-static, GOPI-dynamic and CPPI), the T-bills, and the Stock index in log scale and the upper panel of Table 3 a performance summary.

Interestingly, we find that although the GOPI-static multiplier is lower than the standard multiplier of the CPPI, the average return of the GOPI strategy exceeds the one of the standard CPPI by more than 4% per annum and has a higher value during most of the period, as observed in the left panel of Figure 3. The growth optimal multiplier estimated with the entire sample is equal to 1.6 while the standard multiplier of the CPPI is equal to 3.4[15]. The dynamic version of the optimal multiplier is lower than the standard multiplier about 70% of the months in the sample. By contrast, the CPPI strategy happens to lose almost all of its Cushion at the beginning of the sample and stays very close to its floor value during the rest of the period in spite of having a higher multiplier. On the other hand the GOPI strategy "survives" the periods of high volatility.

The portfolio with the highest cumulative performance during the entire sample is, unsurprisingly, the GOP. On the other hand, the GOP does not respect the constraint that the GOPI and CPPI strategies have and presents a significantly riskier profile. Due to a leveraged position of 160% implied by this pair of assets, the GOP in this case is even more risky than the stock index itself (with a maximum drawdown of 95.6%).

As a second Backtest, we perform an out-of-sample exercise in which we split the sample in two. We use the first half of the sample (i.e., $1926:01-1968:05$) to estimate the parameters of the strategies (multipliers and allocation of the GOP) and use the second half to test the four strategies (i.e., $1968:06-2010:12$).

Table 3: The upper panel presents the results of the in-sample test, for which the parameters were estimated using data of the entire sample period. The lower panel presents the results of the out-of-sample Backtest in which the parameters were estimated using only information available at every point in time. Return stands for the annualized geometric return average, Min represents the minimum value ever attained by the portfolio for an initial value of 100 dollars of each strategy, Vol is the annualized standard deviation of returns and MDD stands for Maximum drawdown.

Period: 01/1926-12/2010	GOP	GOPI(t)	GOPI	CPPI	Cash	Stocks
Return	11.27	9.37	7.88	3.58	3.71	9.62
Vol	29.50	13.84	12.73	4.60	0.88	18.86
MDD	95.56	51.25	51.45	48.46	0.00	83.72
Min	17.00	98.86	99.23	97.49	100.00	41.27
Period: 06/1968-12/2010	GOP	GOPI(t)	GOPI	CPPI	Cash	Stocks
Return	10.41	7.12	6.76	5.63	5.66	9.49
Vol	27.03	8.08	6.63	6.01	0.87	16.16
MDD	72.20	39.66	34.73	29.26	0.00	51.45
Min	39.17	99.66	100.00	99.53	100.00	70.28

We use a long period of time to estimate the parameters in order to avoid the sample bias mostly present in expected return estimates. The result of this second Backtest is presented in the lower panel of Table 3 and the conclusions are similar to the former Backtest. Although the static optimal multiplier (1.7) is again lower than the standard multiplier (3.4) the growth optimal portfolio insurance strategy attains a higher value than the standard CPPI and remains higher during most of the sample period, as illustrated in the right panel of Figure 3. The average return of the GOPI strategy in this sample exceeds the one of the standard CPPI by about 1.5% (dynamic) and 1.13% (static) per annum.

The dynamic version of the optimal multiplier is again lower than the standard multiplier about 70% of the months in the sample, although it attains levels as high as around 8 during low volatility periods.

The GOP attains the highest value among all the strategies but also a minimum value of around 39.2 dollars, compared to minimum values of 100 and 99.5 dollars for the GOPI and CPPI strategies.

Conclusion

We derive the growth rate of the portfolio insurance strategy in the general case with a stochastic reserve asset as a function of the underlying assets' characteristics (i.e. expected returns, volatilities and correlation). The analysis of the portfolio insurance's growth rate illustrates the largely disregarded role that the correlation between the underlying assets plays on the value of this type of strategy. We find that a higher correlation increases the compounded return of the total portfolio.

The growth rate has a close relationship with the value of the portfolio insurance strategy even when common short-selling and leverage constraints are imposed. We also illustrate how the growth rate of the strategy can be decomposed to nail down the relative importance and effect of the characteristics of the underlying assets on the performance of the strategy.

Finally, we introduce the growth optimal portfolio insurance strategy (GOPI), which combines the pragmatic objective of maximizing the growth rate of the portfolio and the common risk-management constraint of preserving a given fraction of the value of the portfolio indexed to a given benchmark. The GOPI is determined by the growth optimal multiplier, while the investor chooses the benchmark and the fraction of value to insure, as in the standard CPPI case. Empirical tests suggest that the GOPI strategy outperforms the standard parametrization of the CPPI over long horizons. The level of the optimal multiplier is most of the time lower than the standard one although it may presents important variations and higher values during low volatility periods.

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Notes

[1]See Platen (2005) and Christensen et al. (2005) for a complete review of the role of the GOP in finance today.

[2]Articles such as Markowitz (1976), Breiman (1961), Luenberger (1998), Long et al. (1990) arguing in favor of the GOP believe that growth optimality is a reasonable investment objective in itself for long horizon investors. Criticisms of the GOP, such as Samuelson (1963) and Ophir (1978), stem from the view that the only rational approach to portfolio selection is to maximize expected utility. The concept of utility based portfolio selection, although widely used, has been criticized by the observation that investors may be unaware of their own utility functions or behave in manners that would be in strong contrast with its predictions (see for instance Bossaerts (2002)).

[3]This effect in fixed-mix portfolios is sometimes called "volatility pumping".

[4]There are at least three practical applications for which portfolio insurance strategies with a reserve asset different from cash are crucial. First, from a pension fund perspective, holding too much cash can be very risky because its liabilities have typically long duration and/or could be indexed to inflation. In this case, the reserve asset is usually defined as a portfolio composed of fixed-income securities trying to match the obligations of the fund. Second, individual investors might want to insure defined long term benefits, bequest objectives and/or a stream of future consumption needs (see Amenc et al. (2009)). This objective can be addressed in the construction of the reserve asset in a similar way as for the pension fund case by hedging the future cash-flow needs as liabilities. Third, asset managers might be given the objective of outperforming a particular Benchmark. One way to comply with this relative performance objective is to define the Core asset as the (presumably) stochastic Benchmark so that the possible underperformance of the portfolio with respect to the Benchmark is limited to a well defined level, while still allowing for increasing upside potential coming from available risk premia and active manager views (for a detailed explanation of the practical advantages of this approach called Dynamic Core-Satellite allocation, see Amenc et al. (2004)).

[5]In the particular case with the riskless reserve asset, this expression simplifies and depends only on the volatility of the risky asset. Hence Black and Perold (1992) calls a discrete-time equivalent of this term the "volatility cost".

[6]The order of the Assets' covariance and the PS asset's variance coefficients is inverted for $1 \leq m < 2$, where 2 is the solution to the equation $2(m-1) = m$. Similarly, the order of expected outperformance and PS asset's variance's coefficients is inverted for $m > \frac{4+\sqrt{12}}{2} = 3.7321$, which is a solution to the quadratic equation $m = \frac{1}{2}(m-1)^2$.

[7]The data and a complete description of it is available at EDHEC-Risk Institute's website, www.edhec-risk.com.

[8]This is equivalent to a δ move in the Index Ratio (see Black and Perold (1992)).

[9]Basak (2002) solves the CRRA utility maximization in the presence of inter-temporal consumption and includes the floor violation restriction embedded in the investor's utility function. The investor's marginal utility smoothly converges towards infinity as opposed to imposing an exogenous constraint in which marginal utility jumps in a discrete way when wealth hits the floor.

[10]Former optimal multipliers do not take into account any possible correlation between the reserve and the risky asset as they use the risk-free asset with constant rate.

[11]Data available from Kenneth French's website.

[12]We use Kevin Sheppard's UCSD Multivariate GARCH toolbox for estimating the DCC model.

[13] We set a minimum value of 1 for the dynamic multiplier. Thus when the growth-optimal formula implies a multiplier lower than 1, the GOPI-dynamic strategy is implemented with a multiplier of 1. The properties of Portfolio Insurance strategies described in this paper change for $m < 1$.

[14] For the pair of assets in question the GOP formula yields a portfolio with leverage and short selling constraints. In the GOP literature, the portfolio has the properties mentioned in the introduction only if all the weights happen to be positive. Thus in this case, what we call the GOP corresponds more precisely to the Kelly criterion.

[15] In this case we use monthly data; hence, the worst case scenario is given by the worst stock return that happened in the 1930s of almost -30%.

Appendix

A Derivation of Portfolio Insurance's Growth Rate

Using the Black-Scholes model for the dynamics of the Reserve (R) and performance seeking asset (S) and the fact that the CPPI can be decomposed into the Floor process, $F_t = kR_t$ and a leveraged Fixed-Mix Portfolio, we now derive an approximation of the growth rate of the CPPI, in the case with a stochastic reserve asset.

Using the interpretation of the Cushion as a Fixed-Mix portfolio with weights m and $1-m$ and result (9) for two assets, the value of the Portfolio Insurance strategy is given by

$$V_m^{PI}(t) = kV_0\left(\frac{R_t}{R_0}\right) + (1-k)V_0\left(\frac{S_t}{S_0}\right)^m\left(\frac{R_t}{R_0}\right)^{1-m} e^{\gamma_m^* t}. \tag{23}$$

The second term in equation (23) is equal to the value of a fixed mix portfolio, thus it can be expressed in terms of its growth rate, as in equation (13),

$$\begin{aligned} C_m(t) &= (1-k)V_0 e^{(\gamma_m^* + m\gamma_S + (1-m)\gamma_R)t + m\sigma_S W_S(t) + (1-m)\sigma_R W_R(t)} && (24)\\ &= (1-k)V_m^{FM}(t) && (25) \end{aligned}$$

Since $C_0 = (1-k)V_0$, by definition of the growth rate (eq. (4) or (5)), the Cushion process's growth rate is

$$\gamma_m^{cushion} = \gamma_m^* + m(\gamma_S - \gamma_R) + \gamma_R, \tag{26}$$

where $\gamma_m^* = \frac{1}{2}m(1-m)\left(\sigma_S^2 + \sigma_R^2 - 2\rho\sigma_S\sigma_S\right)$. Notice that the brownian motion terms in equation (24) disappear after dividing by t as $t \to \infty$, because $W(t) = \sqrt{t}z(t)$ for $z \sim \mathcal{N}(0,1)$. Alternatively, replacing (24) in the other definition of the growth rate, i.e. equation (5), leads to the same result because $E[W(t)] = 0$.

By definition of the growth rate (equation (5)) and using result (23), the portfolio insurance' growth rate can be written as

$$\gamma_m^{PI} = \frac{1}{t}E\left[\ln\left(\frac{kR(t) + (1-k)V_m^{FM}(t)}{V_0}\right)\right]. \tag{27}$$

From Jensen's inequality and the concavity of the log it follows that

$$\gamma_m^{PI} = \frac{1}{t}E\left[\ln\left(\frac{kR(t) + (1-k)V_m^{FM}(t)}{V_0}\right)\right] \geq \frac{1}{t}E\left[k\ln\left(\frac{R(t)}{V_0}\right) + (1-k)\ln\left(\frac{V_m^{FM}(t)}{V_0}\right)\right],$$

by definition of the growth rate we have

$$\gamma_m^{PI} \geq k\gamma_R + (1-k)\gamma_m^{cushion}. \tag{28}$$

Inequality (28) implies

$$\gamma_m^{PI} = k\gamma_R + (1-k)\gamma_m^{cushion} + \phi, \text{ with } \phi \geq 0. \tag{29}$$

Since the growth rate of the Floor process is equal to the growth rate of the Reserve asset, γ_R, equation (29) implies that the growth rate of the Portfolio Insurance strategy is the weighted average of the growth rate of the Reserve asset and the growth rate of the Cushion with weights equal to k and $(1-k)$ respectively plus a positive term ϕ. The latter term can be estimated using a Taylor expansion of the log in equation (27) where all terms above the quadratic are omitted, yielding

$$\phi = \frac{1}{2}k(1-k)\left(\gamma_R^2 + (\gamma_m^{cushion})^2 - 2\gamma_R\gamma_m^{cushion}\right). \tag{30}$$

The log of a weighted average of exponentials can be approximated using the following Taylor expansion where all terms above the quadratic are omitted:

$$\ln(w' \exp(x)) \approx x'w + \frac{1}{2}x'Vx \tag{31}$$

where $V_{i,j} = -w_i w_j \forall i \neq j$, $V_{i,i} = w_i(1-w_i)$, and w is a vector with the same size of vector x, satisfying $\sum_i w_i = 1$. Thus, the "weights" of the weighted average of the exponentials are k and $1-k$ when applying approximation (31) in equation (27).

Let γ_R and $\gamma_m^{cushion}$ be positive and $\gamma_m^{cushion} \geq \gamma_R$. The latter assumption seems reasonable, as investing in a CPPI strategy in which the expected long term return (growth rate) of the Cushion is lower than the growth rate of the reserve asset would not make sense from an economic standpoint (it would better to simply invest in the reserve asset). Under these assumptions, ϕ, as given by equation (30) is a strictly increasing function of $\gamma_m^{cushion}$, thus implying a positive relationship between $\gamma_m^{cushion}$ and γ_m^{PI}.

B Portfolio Insurance's Growth Rate Maximization

As shown in appendix A, the growth rate of the CPPI can be approximated as

$$\gamma_m^{PI} = k\gamma_R + (1-k)\gamma_m^{cushion} + \phi, \tag{32}$$

where

$$\phi = \frac{1}{2}k(1-k)\left(\gamma_R^2 + (\gamma_m^{cushion})^2 - 2\gamma_R\gamma_m^{cushion}\right). \tag{33}$$

Deriving equation (32) with respect to m and equating to zero yields three roots, one of which is the maximum of the function in the range where $\gamma_m^{cushion} \geq \gamma_R$. The latter assumption seems reasonable from an economic standpoint: investing in a CPPI strategy in which the expected long term return (growth rate) of the Cushion is lower than the growth rate of the reserve asset would guarantee to have a portfolio with a lower value than the reserve asset in the long run (i.e. it would be better to simply invest in the reserve asset).

Outside the range $\gamma_m^{cushion} \geq \gamma_R$, our approximation of γ_m^{PI} is strictly increasing for very high values of m (i.e. without range limits there approximation has no maximum, but only a local maximum). However, the second order approximation of the log used is less accurate for higher (absolute) values of the multiplier. For this reason, in order to maximize γ_m^{PI}, we restrict the function to positive values of γ_R and assume that $\gamma_m^{cushion} \geq \gamma_R$. In Figure 4 we make an illustration with the following parameters: $\mu_S = 0.08$, $\sigma_S = 0.15$, $\mu_S = 0.03$, $\sigma_S = 0.05$ and $\rho = 0.3$.

Under these assumptions, maximizing (32) yields the *growth optimal* multiplier:

$$m^* = \frac{\gamma_S - \gamma_R + \gamma^*}{2\gamma^*},$$

where $\gamma^* = \frac{1}{2}(\sigma_S^2 + \sigma_R^2 - 2\rho\sigma_S\sigma_R)$. This multiplier also maximizes the growth rate of the Cushion.

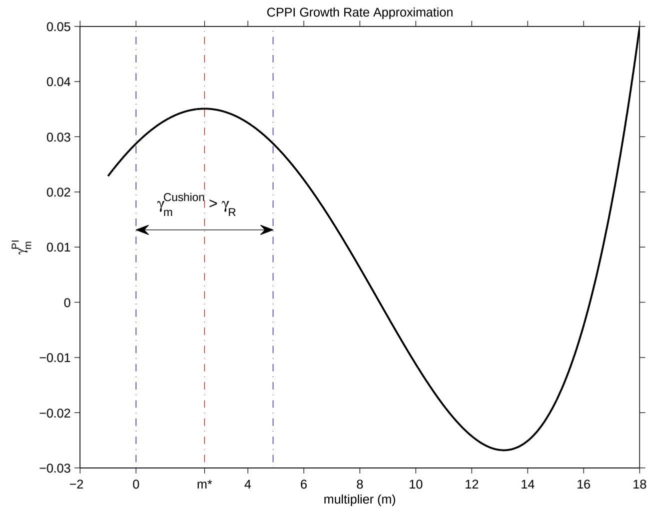

Figure 4: The assets parameters used for this illustration are: $\mu_S = 0.08$, $\sigma_S = 0.15$, $\mu_S = 0.03$, $\sigma_S = 0.05$ and $\rho = 0.3$, we find that the function attains higher values than the local maximum for $m > 16$. Although our approximation of γ_m^{PI} is strictly increasing for very high values of m, the second order approximation of the log used is less accurate for higher values of the multiplier. The range over which $\gamma_m^{cushion} \geq \gamma_R$ is pointed with the double arrow.

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